Closing Wed: HW_8A, 8B (8.3, 9.1)
Closing Next Wed: HW_9A, 9B (9.3, 9.4)
Midterm 2 will be returned Tuesday.

9.1 Intro to Differential Equations

A **differential equation** is an equation involving derivatives.

A **solution to a differential equation** is any function that satisfies the equation.

Here is an example of a differential equation which you already know how to solve.

Entry Task:

Find $y = y(x)$ such that

$$\frac{dy}{dx} - 8x = x^2 \text{ and } y(0) = 5.$$

Check your final answer

Example

Consider the differential equation:

$$\frac{dP}{dt} = 2P$$

(a) Is $P(t) = 8e^{2t}$ a solution?

(b) Is $P(t) = t^3$ a solution?

(c) Is $P(t) = 0$ a solution?

The **general solution** to

$$\frac{dP}{dt} = 2P$$

is

$$P(t) = Ce^{2t},$$

for any constant C.

We will learn how to find this next time.

Example: Consider the 2nd order differential equation

$$y'' + 2y' + y = 0.$$

(a) Is $y = e^{-2t}$ a solution?

(b) Is $y = t\,e^{-t}$ a solution?

(c) There is a sol'n that looks like

$$y = e^{rt}.$$

Can you find the value of *r* that works?

Application Notes:

$\frac{dy}{dt}$ = "instantaneous **rate of change** of y with respect to t"

"A is proportional to B" means

A = kB, where k is a constant.

In other words, A/B = k.

An Example:

A common assumption for melting snow/ice is the rate at which the object is melting (rate of change of volume) is proportional to the exposed surface area.

Consider a melting snowball:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Write down the differential equation for *r*.

Four applied examples from homework:

1. Natural Unrestricted population

Assumption: "*The rate of growth of a population is proportional to the size of the population*."

P(t) = the population at year *t*,

$\frac{dP}{dt} =$ the rate of change of the population with respect to time (i.e. rate of growth).

So the assumption is equivalent to the differential equation

$$\frac{dP}{dt} = kP,$$

for some constant *k*

(we call *k* the *relative growth rate*)

2. Newton's Law of Cooling

Assumption: "*The rate of cooling is proportional to the temperature difference between the object and its surroundings*."

$T_s =$ constant temp. of the surroundings

$T(t) =$ the temp. of the object at time *t*,

$\frac{dT}{dt} =$ the rate of change of the temp with respect to time (*i.e.* rate of cooling).

$T - T_s =$ temp. difference between object and surroundings.

So Newton's Law of Cooling is equivalent to the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

for some constant *k* (cooling constant).

3. A Mixing Problem

Assume a 50 gallon vat is initially full of pure water. A salt water mixture is being dumped **into** the vat at 2 gal/min and this mixture contains 3 g/gal. The vat is mixed together.

At the same time, the mixture is coming **out** of the vat at 2 gal/min.

Let y(t) = grams of salt in vat at time *t*.

$\frac{y(t)}{50}$ = salt per gallon in vat at time, *t*.

$\frac{dy}{dt} =$ the rate (g/min) at which salt is changing with respect to time.

RATE IN = (3 g/gal)(2 gal/min) = 6 g/min

RATE OUT = ($\frac{y}{50}$ g/gal)(2gal/min) = $\frac{y}{25}$ g/min

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25}$$

4. All motion problems!

Consider an object of mass *m* kg moving up and down on a straight line.

Let y(t) = `height at time *t*'

$\frac{dy}{dt} =$ `velocity at time *t*'

$\frac{d^2y}{dt^2} =$ `acceleration at time *t*'

Newton's 2nd Law says:

(mass)(acceleration) = Force

$$m\frac{d^2y}{dt^2} = \text{sum of forces on the object}$$

Only taking into account gravity the differential equation is:

$$m\frac{d^2y}{dt^2} = -mg$$

Consider gravity and air resistance (assuming the force due to air resistance is proportional to velocity) the differential equation is:

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$